

Introduction to Perturbative QCD

(An Introduction/Historical Review)

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OUTLINE

- 1. Introduction: From the quark model to QCD
- 2. Self-consistency: antiquarks in hadron-hadron scattering
- 3. Factorization and Evolution
- 4. How we get away with pQCD: IR safety, factorize, evolve, resum
- 5. Inclusive annihilation in pQCD
- 6. Using pQCD Corrections
- 7. Getting PDFs from the data
- 8. Using resummation: the Q_T distribution
- 9. Putting it all together: pions and jets in hadronic collisions

1. INTRODUCTION: FROM QUARKS TO QCD

- Spectroscopy and the quark model
 - The discovery of quarks: qqq and $\bar{q}q$ with $q = u, d, s$ generate observed spectrum of baryons and mesons
 - Decay of $\bar{s}s$ states to K, \bar{K} states (OZI rule) indicates continuity of quark lines
 - Non-relativistic wave functions predict ratios of magnetic moments μ_n/μ_p etc.

- **Dynamical evidence: form factors & structure functions**

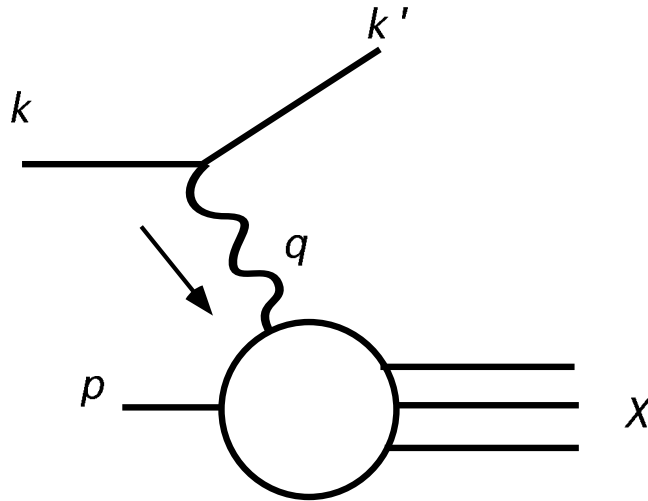
- **Form factors: $ep \rightarrow ep$ elastic**

$$\frac{d\sigma}{d\Omega_e} = \left[\frac{\alpha_{\text{EM}}^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \right] \frac{E'}{E} \left(\frac{|G_E(Q)|^2 + \tau |G_M(Q)|^2}{1 + \tau} + 2\tau |G_M(Q)|^2 \tan^2 \theta/2 \right)$$

- **schematically:**

$$\frac{d\sigma_{ep \rightarrow ep}(Q)}{dQ^2} \sim \frac{d\sigma_{ee \rightarrow ee}(Q)}{dQ^2} \times G(Q) \quad \text{with} \quad G(Q) \sim \frac{1}{\left(1 + \frac{Q^2}{\mu_0^2}\right)^2}$$

- **Structure functions: ep inclusive, unpolarized, p rest frame**



$$\frac{d\sigma}{dE' d\Omega} = \left[\frac{\alpha_{\text{EM}}^2}{2SE \sin^4(\theta/2)} \right] \left(2 \sin^2(\theta/2) F_1(x, Q^2) + \frac{m \cos^2(\theta/2)}{E - E'} F_2(x, Q^2) \right)$$

with $x = \frac{Q^2}{2p_N \cdot q}$

– **More generally, with spin,** $\sigma \sim (leptonic)_{\mu\nu} W^{\mu\nu}$,

$$\begin{aligned}
 W^{\mu\nu} &= \frac{1}{4\pi} \int d^4z e^{iq \cdot z} \langle P, S | J^\mu(z) J^\nu(0) | P, S \rangle \\
 &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1(x, Q^2) \\
 &\quad + \left(P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) \left(P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) F_2(x, Q^2) \\
 &\quad + iM_N \epsilon^{\mu\nu\rho\sigma} q_\sigma \left[\frac{S_\sigma}{P \cdot q} g_1(x, Q^2) + \frac{S_\sigma(P \cdot q) - P_\sigma(S \cdot q)}{(P \cdot q)^2} g_2(x, Q^2) \right]
 \end{aligned}$$

- **Scaling:** $F_2(x, Q^2) \sim F_2(x) \Rightarrow$ **Point-like, quasi-free scattering**
- $F_2 \sim 2xF_1$: **Spin-1/2**
- **Parton model structure functions**

$$F_{2,N}(x) = \sum_q e_q^2 x q_N(x)$$

$$g_{1,N}(x) = \frac{1}{2} \sum_q e_q^2 (\Delta q_N(x) + \Delta \bar{q}_N(x))$$

- **Notation:** $f_{q/N}(x) = q_N(x)$ etc. **Probability for struck quark q to have momentum fraction x .**
- **Notation:** $\Delta q_N = q_N^+ - q_N^-$ **with $q^\pm(x)$ probability for struck quark q to have momentum fraction x and helicity with (+) or against (-) N helicity.**

- At the same time, a quark model paradox \Rightarrow color
 - First of all, nobody had *seen* a quark (confinement), but also
 - A problem with the quark model: quarks have spin-1/2 but nucleon quark model wave function was symmetric
- But spin-1/2 particles are all fermions – right?
- Fast-forward resolution:
 - Han, Nambu 1965: quarks come in 3 triplets of colors
 - Quarks in baryons are antisymmetric in quantum number of the group SU(3)

- The birth of QCD: SU(3)

- A nonabelian gauge theory built on color ($q = q_1 q_2 q_3$):

$$\mathcal{L}_{QCD} = \sum_q \bar{q} (i\not{D} - g_s \not{A} + m_q) q - \frac{1}{4} F_{\mu\nu}^2[A]$$

- Think of: $\mathcal{L}_{EM} = K_e + J_{EM} \cdot A + (E^2 - B^2)$
- The Yang-Mills gauge theory of quarks (q) and gluons (A)
Gluons: like “charged photons”. The field a source for itself.
- Just the right currents to couple to EM and Weak AND . . .

- Just the right kind of forces: **QCD charge is “antishielded”**
and *grows with distance*

$b_0 = 11 - 2n_{\text{quarks}}/3$ we get:

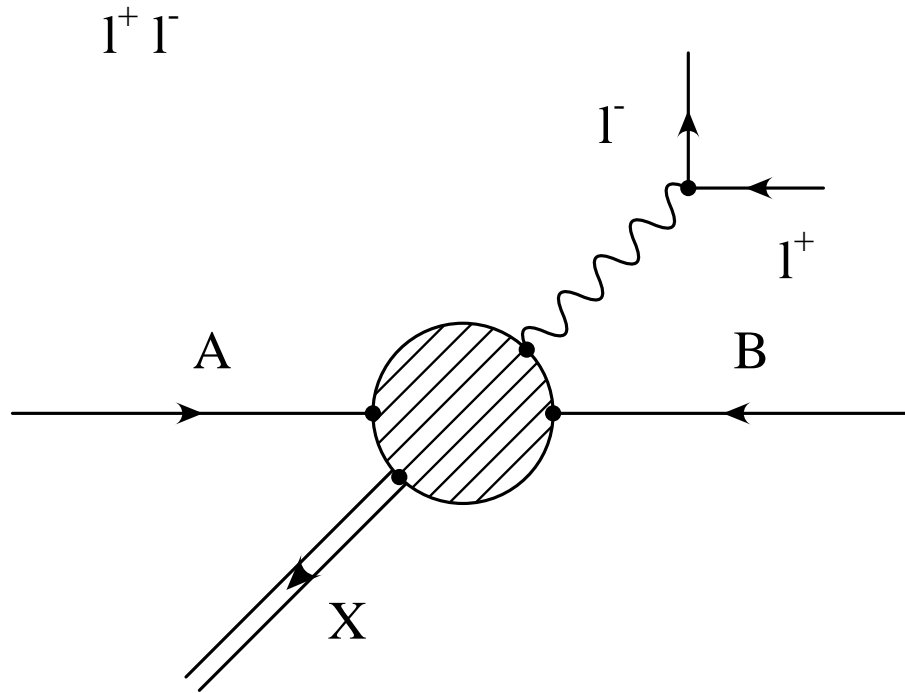
$$\alpha_s(\mu') = \frac{g_s^2}{4\pi} = \frac{\alpha_s(\mu)}{1 + b_0 \frac{\alpha_s(\mu)}{4\pi} \ln \left(\frac{\mu'}{\mu} \right)^2} = \frac{4\pi}{b_0 \ln(Q^2/\Lambda_{\text{QCD}}^2)}$$

Quantum field theory: every state with the same quantum nos. as uud in the proton . . . is present at least some of the time

So antiquarks are in the nucleon: $uudd\bar{d}$, etc.

What it means: $q\bar{q}$ annihilation processes in NN collisions as d, u from one nucleon collides with \bar{d}, \bar{u} from another

Annihilation into what? Back to quarks, & gluons, yes, but also



γ , W , Z , H . . .

Which brings us to . . .

- The rest of the standard model: $SU(3) \times SU(2)_L \times U(1)$
- Quark and lepton fields: L and R
 - $\psi = \psi^{(L)} + \psi^{(R)} = \frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\psi$; $\psi = q, \ell$
 - **Helicity: spin along \vec{p} (R=right handed) or opposite (L=left handed) in solutions to Dirac equation**
 - $\psi^{(L)}$: only L particle solutions; but R antiparticle solutions
 - $\psi^{(R)}$: only R particle solutions, L antiparticle

$$\begin{array}{ccc}
 q_i^{(L)} & = & \left(u_i, d'_i = V_{ij} d_j \right) \quad u_i^{(R)}, d_i^{(R)} \\
 (u, d') & & (c, s') \quad (t, b') \\
 \ell_i^{(L)} & = & \left(e_i, \nu_i \right) \quad e_i^{(R)}, \nu_i^{(R)} \\
 (\nu_e, e) & & (\nu_\mu, \mu) \quad (\nu_\tau, \tau)
 \end{array}$$

- V_{ij} is the “CKM” matrix

- Weak vector bosons: electroweak gauge groups
 - **SU(2): three vector bosons B_i , coupling g**
 - **U(1); one vector boson C , coupling g'**
 - The physical bosons:

$$W^\pm = B_1 \pm iB_2$$

$$Z = -C \sin \theta_W + B_3 \cos \theta_W$$

$$\gamma \equiv A = C \cos \theta_W + B_3 \sin \theta_W$$

$$\sin \theta_W = g' / \sqrt{g^2 + g'^2} \qquad M_W = M_Z / \cos \theta_W$$

$$e = gg' / \sqrt{g^2 + g'^2} \qquad M_W \sim g / \sqrt{G_F}$$

- The interactions of quarks and leptons with the photon, W, Z

$$\begin{aligned}
\mathcal{L}_{\text{EW}}^{(fermion)} = & \sum_{\text{all } \psi} \bar{\psi} (i\not{D} - e\lambda_{\psi}\not{A} - (gm_{\psi}2M_W)h) \psi \\
& - (g/\sqrt{2}) \sum_{q_i, e_i} \bar{\psi}^{(L)} (\sigma^+ \not{W}^+ + \sigma^- \not{W}^-) \psi^{(L)} \\
& - (g/2 \cos \theta_W) \sum_{\text{all } \psi} \bar{\psi} (v_f - a_f \gamma_5) \not{Z} \psi
\end{aligned}$$

- Interactions with the Higgs $h \propto \text{mass}$
 - Interactions with W are through ψ_L 's only
 - Neutrino Z exchange is sensitive to $\sin^2 \theta_W$, even at low energy
- Observation made it clear by early 1970's that $M_W \sim g/\sqrt{G_F}$ is large (need for colliders)

- **Symmetry violations in the standard model**
 - W 's interact through $\psi^{(L)}$ **only** $\psi = q, \ell$
 - Left-handed quarks, leptons; right-handed antiquarks, leptons
 - Parity (P) exchanges L/R; Charge conjugation (C) exchanges particles, antiparticles
 - CP combination OK $L \rightarrow R \rightarrow L$ if all else equal, but it's not (quite). Complex phases in CKM $V \rightarrow$ CP violation.

2. SELF-CONSISTENCY: ANTIQUARKS IN HADRON HADRON SCATTERING

- The Inclusive Drell-Yan Cross Section

Parton Model: “Impulse approximation”. The template (1970):

$$\frac{d\sigma_{NN \rightarrow \mu\bar{\mu}+X}(Q, p_1, p_2)}{dQ^2 d\dots} \sim \int d\xi_1 d\xi_2 \sum_{a=q\bar{q}} \frac{d\sigma_{a\bar{a} \rightarrow \mu\bar{\mu}}^{\text{EW, Born}}(Q, \xi_1 p_1, \xi_2 p_2)}{dQ^2 d\dots}$$

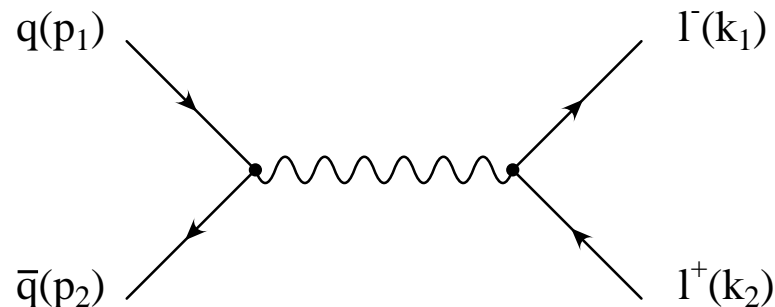
× (probability to find parton $a(\xi_1)$ in N)
× (probability to find parton $\bar{a}(\xi_2)$ in N)

The probabilities are $f_{q/N}(x)$'s from DIS!

Recall how it works (with colored quarks) ...

- **The Born cross section**

$\sigma^{\text{EW,Born}}$ is all from this diagram (ξ 's set to unity):



With this matrix element

$$M = e_q \frac{e^2}{\hat{s}} \bar{u}(k_1) \gamma_\mu v(k_2) \bar{v}(p_2) \gamma^\mu u(p_1)$$

- **First square and sum/average M . Then evaluate phase space.**

- **Total cross section:**

$$\begin{aligned}\sigma_{q\bar{q} \rightarrow \mu\bar{\mu}}^{\text{EW, Born}}(x_1 p_1, x_2 p_2) &= \frac{1}{2\hat{s}} \int \frac{d\Omega}{32\pi^2} \frac{e_q^2 e^4}{\textcolor{red}{3}} (1 + \cos^2 \theta) \\ &= \frac{4\pi\alpha^2}{\textcolor{red}{9}M^2} \sum_q e_q^2 \equiv \sigma_0(M)\end{aligned}$$

With M the pair mass **and 3 for color average**

Now we're ready for the parton model differential cross section for NN scattering:

Pair mass (M) and rapidity

$$\eta \equiv (1/2) \ln(Q^+/Q^-) = (1/2) \ln[(Q^0 + Q^3)/(Q^0 - Q^3)]$$

overdetermined \rightarrow delta functions in the Born cross section

$$\begin{aligned}
\frac{d\sigma_{NN \rightarrow \mu\bar{\mu}+X}^{(PM)}(Q, p_1, p_2)}{dM^2 d\eta} &= \int_{\xi_1, \xi_2} \sum_{a=q\bar{q}} \sigma_{a\bar{a} \rightarrow \mu\bar{\mu}}^{\text{EW, Born}}(\xi_1 p_1, \xi_2 p_2) \\
&\times \delta(M^2 - \xi_1 \xi_2 S) \delta\left(\eta - \frac{1}{2} \ln\left(\frac{\xi_1}{\xi_2}\right)\right) \\
&\times f_{a/N}(\xi_1) f_{\bar{a}/N}(\xi_2)
\end{aligned}$$

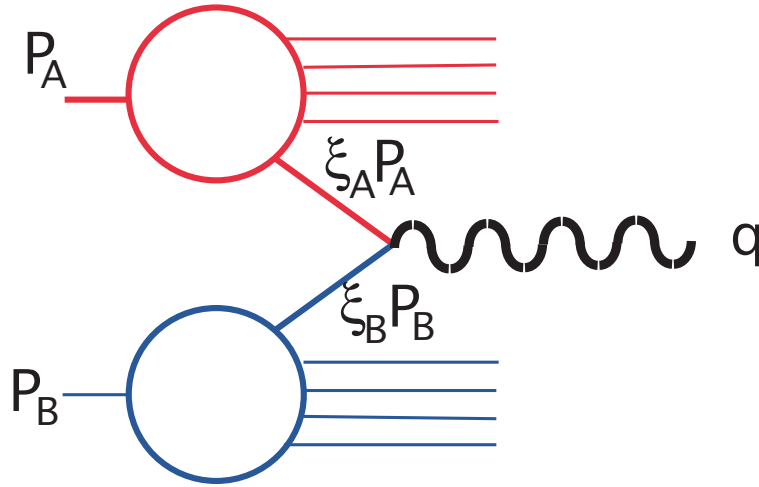
and integrating over rapidity,

$$\frac{d\sigma}{dM^2} = \left(\frac{4\pi\alpha_{\text{EM}}^2}{9M^4} \right) \int_0^1 d\xi_1 d\xi_2 \delta(\xi_1 \xi_2 - \tau) \sum_a \lambda_a^2 f_{a/N}(\xi_1) f_{\bar{a}/N}(\xi_s)$$

Drell and Yan, 1970 (aside from 1/3 for color)

Analog of DIS: scaling in $\tau = Q^2/S$

- The parton model picture



- All QCD radiation in the f 's . . . but why?
- Asymptotic freedom has something to do with this . . . but how? What to do in QFT?

3. FACTORIZATION AND EVOLUTION

**Factorization as a generalization
of the Parton Model**

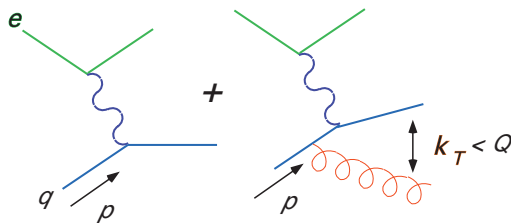
Evolution

FACTORIZATION AS A GENERALIZATION OF THE PARTON MODEL

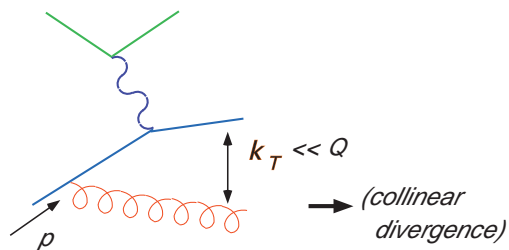
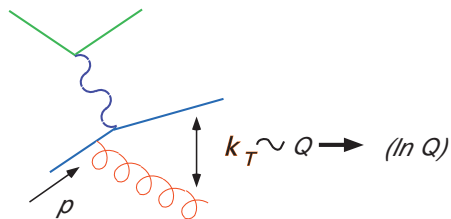
- Challenge: use AF in observables
(cross sections (σ) (also some amplitudes . . .))
that are *not infrared safe*
- Possible *if*: σ has a short-distance subprocess.
Separate *IR Safe* from **IR**: **this is factorization**
- **IR Safe** part (short-distance) is **calculable in pQCD**
- Infrared part – **example: parton distribution** –
measurable and universal
- Infrared safe – insensitive to soft gluon emission
collinear rearrangements

- Just like Parton Model except in Parton Model the infrared safe part is $\sigma_{\text{Born}} \Rightarrow f(x)$ **normalized uniquely**
- In pQCD must define parton distributions more carefully: **the factorization scheme**

Basic observation: virtual states not truly frozen.
Some states fluctuate on scale $1/Q$:



Long-lived states \Rightarrow Collinear Logs (IR)



Short-lived states $\Rightarrow \ln(Q)$

RESULT: FACTORIZED DIS

$$\begin{aligned} F_2^{\gamma q}(x, Q^2) &= \int_x^1 d\xi \, C_2^{\gamma q} \left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu) \right) \\ &\quad \times f_{q/q}(\xi, \mu_F, \alpha_s(\mu)) \\ &\equiv C_2^{\gamma q} \left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu) \right) \otimes f_{q/q}(\xi, \mu_F, \alpha_s(\mu)) \end{aligned}$$

- f **has** $\ln(\mu_F/\Lambda_{\text{QCD}})$. . .
- C **has** $\ln(Q/\mu), \ln(\mu_F/\mu)$
- **Often pick** $\mu = \mu_F$ **and often pick** $\mu_F = Q$. **So often see:**

$$F_2^{\gamma q}(x, Q^2) = C_2^{\gamma q} \left(\frac{x}{\xi}, \alpha_s(Q) \right) \otimes f_{q/q}(\xi, Q^2)$$

- But we still need to specify what we *really* mean by factorization: *scheme* as well as *scale*
- For this, compute $F_2^{\gamma q}(x, Q)$
- Keep $\mu = \mu_F$ for simplicity

- “Compute quark-photon scattering” – *What does this mean?*
 - Must use an *IR-regulated* theory
 - Extract the *IR Safe part* *then* take away the regularization
 - *Let’s see how it works . . .*
 - **At** *zeroth order – no interactions:*
 - $C^{\gamma q_f(0)} = Q_f^2 \delta(1 - x/\xi)$
(Born cross section; parton model)
 - $f_{q_f/q_{f'}}^{(0)}(\xi) = \delta_{ff'} \delta(1 - \xi)$
(at zeroth order, momentum fraction conserved)

$$\begin{aligned}
F_2^{\gamma q_f^{(0)}}(x, Q^2) &= \int_x^1 d\xi \, C_2^{\gamma q_f^{(0)}}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu)\right) \\
&\quad \times f_{q_f/q_f}^{(0)}(\xi, \mu_F, \alpha_s(\mu)) \\
&= Q_f^2 \int_x^1 d\xi \, \delta(1 - x/\xi) \, \delta(1 - \xi) \\
&= Q_f^2 \, x \, \delta(1 - x)
\end{aligned}$$

– On to one loop . . .

$F^{\gamma q}$ @ AT ONE LOOP: FACTORIZATION SCHEMES

- Start with F_2 for a *quark*:

$$\left| \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} \right|^2 \text{ "real"}$$

$$+ 2 \operatorname{Re} \left(\begin{array}{c} \text{diagram 3} \end{array} \right)^* \left(\begin{array}{c} \text{diagram 4} \\ \text{diagram 5} \end{array} \right)$$

"virtual"

Have to combine final states with different phase space . . .

“Plus Distributions”:

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}$$

$$\int_0^1 dx f(x) \left(\frac{\ln(1-x)}{1-x} \right)_+ \equiv \int_0^1 dx (f(x) - f(1)) \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where

- $f(x)$ will be parton distributions
- $f(x)$ term: real gluon, with momentum fraction $1-x$
- $f(1)$ term: virtual, with elastic kinematics

A Special Distribution

DGLAP “evolution kernel” = “splitting function”

$$P_{qq}^{(1)}(x) = C_F \frac{\alpha_s}{\pi} \left[\frac{1+x^2}{1-x} \right]_+$$

- Will see: P_{qq} a probability per unit $\log k_T$

Expansion and Result:

$$F_2^{\gamma q}(x, Q^2) = \int_x^1 d\xi \, C_2^{\gamma q} \left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu) \right) \\ \times f_{q/q}(\xi, \mu_F, \alpha_s(\mu))$$

$$F_2^{\gamma qf}(x, Q^2) = C_2^{(0)} f^{(0)} \\ + \frac{\alpha_s}{2\pi} C^{(1)} f^{(0)} \\ + \frac{\alpha_s}{2\pi} C^{(0)} f^{(1)} + \dots$$

$$\begin{aligned}
F_2^{\gamma qf}(x, Q^2) &= Q_f^2 \left\{ x \, \delta(1-x) \right. \\
&\quad + \frac{\alpha_s}{2\pi} C_F \left[\frac{1+x^2}{1-x} \left(\frac{\ln(1-x)}{x} \right) + \frac{1}{4} (9-5x) \right]_+ \\
&\quad \left. + \frac{\alpha_s}{2\pi} C_F \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \left[\frac{1+x^2}{1-x} \right]_+ \right\} + \dots
\end{aligned}$$

$$F_1^{\gamma qf}(x, Q^2) = \frac{1}{2x} \left\{ F_2^{\gamma qf}(x, Q^2) - C_F \alpha \frac{\alpha_s}{\pi^2} 2x \right\}$$

Factorization Schemes

$\overline{\text{MS}}$

$$f_{q/q}^{(1)}(x, \mu^2) = \frac{\alpha_s}{\pi^2} P_{qq}(x) \int_0^{\mu^2} \frac{dk_T^2}{k_T^2}$$

With k_T -integral “IR regulated”.

Advantage: technical simplicity; not tied to process.

$C^{(1)}(x)_{\overline{\text{MS}}} = (\alpha_s/2\pi) P_{qq}(x) \ln(Q^2/\mu^2) + \mu$ -independent

DIS:

$$f_{q/q}(x, \mu^2) = \frac{\alpha_s}{\pi^2} F^{\gamma qf}(x, \mu^2)$$

Absorbs all uncertainties in DIS into a PDF.

Closer to experiment for DIS.

$$C^{(1)}(x)_{\overline{\text{DIS}}} = (\alpha_s/2\pi) P_{qq}(x) \ln(Q^2/\mu^2) + 0$$

*Using the Regulated Theory
and
Getting Parton Distributions for Real Hadrons*

- **IR-regulated QCD is not *REAL* QCD**
- *BUT* it only differs at low momenta
- *THUS* we can use it for IR Safe functions: $C_2^{\gamma q}$, etc.
- **This enables us to get PDFs for real hadrons . . .**

- Compute $F_2^{\gamma q}, F_2^{\gamma G} \dots$
- Define factorization scheme; find IR Safe C 's
- Use factorization in the full theory

$$F_2^{\gamma N} = \sum_{a=q_f, \bar{q}_f, G} C^{\gamma a} \otimes f_{a/N}$$

- Measure F_2 ; then use the known C 's to derive $f_{a/N}$
- Multiple flavors and cross sections
complicate technicalities; not logic (Global Fits)

NOW HAVE $f_{a/N}(\xi, \mu^2)$

USE IT IN ANY OTHER PROCESS THAT FACTORIZES

EVOLUTION

- Q^2 -dependence
- In general, Q^2/μ^2 dependence still in $C_a(x/\xi, Q^2/\mu^2, \alpha_s(\mu))$
Choose $\mu = Q$

$$F_2^{\gamma A}(x, Q^2) = \sum_a \int_x^1 d\xi C_2^{\gamma a} \left(\frac{x}{\xi}, 1, \alpha_s(Q) \right) f_{a/A}(\xi, \mu^2)$$

$Q \gg \Lambda_{\text{QCD}} \rightarrow$ compute C 's in PT .

$$C_2^{\gamma a} \left(\frac{x}{\xi}, 1, \alpha_s(Q) \right) = \sum_n \left(\frac{\alpha_s}{\pi} \right)^n C_2^{\gamma a(n)} \left(\frac{x}{\xi} \right)$$

But still need PDFs at $\mu = Q$: $f_{a/A}(\xi, Q^2)$

– Remarkable result: **EVOLUTION**

Can use $f_{a/A}(x, Q_0^2)$ to determine $f_{a/A}(x, Q^2)$ and hence $F_{1,2,3}(x, Q^2)$ for any Q !

So long as $\alpha_s(Q)$ is still small

- Illustrate by a ‘nonsinglet’ distribution

$$F_a^{\gamma\text{NS}} = F_a^{\gamma p} - F_a^{\gamma n}$$

$$F_2^{\gamma\text{NS}}(x, Q^2) = \sum_a \int_x^1 d\xi \, C_2^{\gamma\text{NS}}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)\right) f_{\text{NS}}(\xi, \mu^2)$$

Gluons, antiquarks cancel

At one loop: $C_2^{\text{NS}} = C_2^{\gamma N}$

- ‘Mellin’ Moments and Anomalous Dimensions

$$\bar{f}(N) = \int_0^1 dx \, x^{N-1} f(x)$$

- Reduces convolution to a product

$$f(x) = \int_x^1 dy \, g\left(\frac{x}{y}\right) h(y) \rightarrow \bar{f}(N) = \bar{g}(N) \bar{h}(N+1)$$

- **Moments applied to NS structure function:**

$$\bar{F}_2^{\gamma\text{NS}}(N, Q^2) = \bar{C}_2^{\gamma\text{NS}} \left(N, \frac{Q}{\mu}, \alpha_s(\mu) \right) \bar{f}_{\text{NS}}(N, \mu^2)$$

(Note $f_{\text{NS}}(N, \mu^2) \equiv \int_0^1 d\xi \xi^N f(\xi, \mu^2)$ here.)

- $\bar{F}_2^{\gamma\text{NS}}(N, Q^2)$ is **PHYSICAL**

$$\Rightarrow \mu \frac{d}{d\mu} \bar{F}_2^{\gamma\text{NS}}(N, Q^2) = 0$$

- ‘Separation of variables’

$$\mu \frac{d}{d\mu} \ln \bar{f}_{\text{NS}}(N, \mu^2) = -\gamma_{\text{NS}}(N, \alpha_s(\mu))$$

$$\gamma_{\text{NS}}(N, \alpha_s(\mu)) = \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma_{\text{NS}}} (N, \alpha_s(\mu))$$

- Because α_s is the only variable held in common!

$$\mu \frac{d}{d\mu} \ln \bar{f}_{\text{NS}}(N, \mu^2) = -\gamma_{\text{NS}}(N, \alpha_s(\mu))$$

$$\gamma_{\text{NS}}(N, \alpha_s(\mu)) = \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma_{\text{NS}}}(N, \alpha_s(\mu))$$

- Only need to know C 's $\Rightarrow \gamma_n$ from IR regulated theory!



Q -DEPENDENCE DETERMINED BY PT

EVOLUTION

THIS WAS HOW WE FOUND OUT QCD IS 'RIGHT'

**THIS IS HOW QCD PREDICTS PHYSICS
AT NEW SCALES**

γ_{NS} AT ONE LOOP

Hint: $(1 - x^2)/(1 - x) = 1 + x \dots (1 - x^k)/(1 - x) = \sum_{i=0}^{k-1} x^i$

$$\begin{aligned}
\gamma_{\text{NS}}(N, \alpha_s) &= \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma_{\text{NS}}} (N, \alpha_s(Q)) \\
&= \mu \frac{d}{d\mu} \left\{ (\alpha_s/2\pi) \bar{P}_{qq}(N) \ln(Q^2/\mu^2) + \mu \text{ indep.} \right\} \\
&= -\frac{\alpha_s}{\pi} \int_0^1 dx \, x^{N-1} P_{qq}(x) \\
&= -\frac{\alpha_s}{\pi} C_F \int_0^1 dx \left[(x^{N-1} - 1) \frac{1+x^2}{1-x} \right] \\
&= -\frac{\alpha_s}{\pi} C_F \left[4 \sum_{m=2}^N \frac{1}{m} - 2 \frac{2}{N(N+1)} + 1 \right] \\
&\equiv -\frac{\alpha_s}{\pi} \gamma_{\text{NS}}^{(1)}
\end{aligned}$$

SOLUTION: SCALE BREAKING

$$\mu \frac{d}{d\mu} \bar{f}_{\text{NS}}(N, \mu^2) = -\gamma_{\text{NS}}(N, \alpha_s(\mu)) \bar{f}_{\text{NS}}(N, \mu^2)$$

$$\bar{f}_{\text{NS}}(N, \mu^2) = \bar{f}_{\text{NS}}(N, \mu_0^2) \times \exp \left[-\frac{1}{2} \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_{\text{NS}}(N, \alpha_s(\mu')) \right]$$

\Downarrow

$$\bar{f}_{\text{NS}}(N, Q^2) = \bar{f}_{\text{NS}}(N, Q_0^2) \left(\frac{\ln(Q^2/\Lambda_{\text{QCD}}^2)}{\ln(Q_0^2/\Lambda_{\text{QCD}}^2)} \right)^{-2\gamma_N^{(1)}/\beta_0}$$

Hint:

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{\text{QCD}}^2)}$$

So also:

$$\bar{f}_{\text{NS}}(N, Q^2) = \bar{f}_{\text{NS}}(N, Q_0^2) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{-2\gamma_N^{(1)}/\beta_0}$$

$$\bar{f}_{\text{NS}}(N, Q^2) = \bar{f}_{\text{NS}}(N, Q_0^2) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{-2\gamma_N^{(1)}/\beta_0}$$

- **Mild' scale breaking**
- **For $\alpha_s \rightarrow \alpha_0 \neq 0$, get a power Q -dependence:**

$$(Q^2)^{\gamma^{(1)} \frac{\alpha_s}{2\pi}}$$

- **QCD's consistency with the Parton Model (73-74)**

$$\mu \frac{d}{d\mu} \bar{f}_{\text{NS}}(N, \mu^2) = -\gamma_N(\alpha_s(\mu)) \bar{f}_{\text{NS}}(N, \mu^2)$$

\Downarrow

$$\mu \frac{d}{d\mu} \bar{f}_{\text{NS}}(N, \mu^2) = \int_x^1 \frac{d\xi}{\xi} P_{\text{NS}}(\xi, \alpha_s(\mu)) \bar{f}_{\text{NS}}(\xi, \mu^2)$$

Splitting function \leftrightarrow Moments

$$\int_0^1 dx \, x^{N-1} P_{qq}(x, \alpha_s) = \gamma_{qq}(N, \alpha_s)$$

BEYOND NONSINGLET COUPLED EVOLUTION

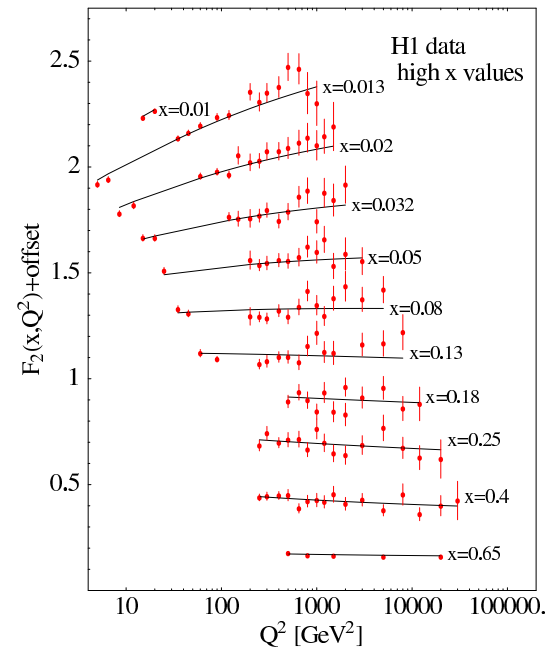
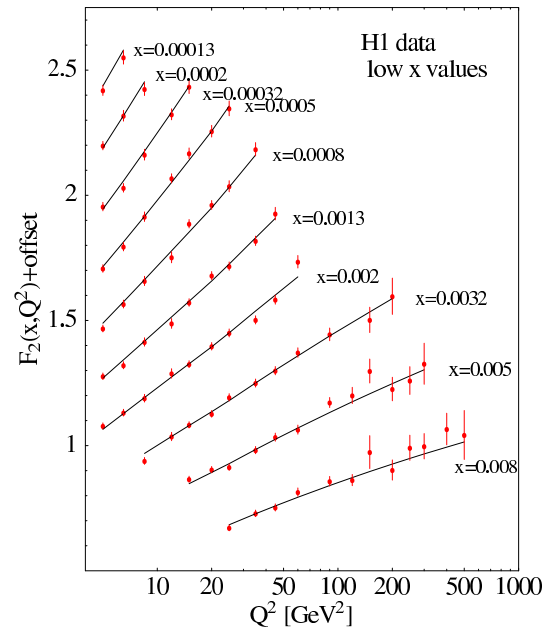
$$\mu \frac{d}{d\mu} \bar{f}_{b/A}(N, \mu^2) = \sum_{b=q, \bar{q}, G} \int_x^1 \frac{d\xi}{\xi} P_{ab}(\xi, \alpha_s(\mu)) \bar{f}_{b/A}(\xi, \mu^2)$$

Physical Contxt of Evolution

- Parton Model: $f_{a/A}(x)$ density of parton a with momentum fraction x , assumed independent of Q
- PQCD: $f_{a/A}(x, \mu)$: same density, but with transverse momentum $\leq \mu$

- If there *were* a maximum transverse momentum Q_0 , $f(x, Q_0)$ would freeze for $\mu \geq Q_0$
- *Not so* in renormalized PT
- Scale breaking measures the change in the density as maximum transverse momentum increases
- Cross sections we compute still depend on our choice of μ through uncomputed “higher orders” in C and evolution

– Evolution in DIS (with CTEQ6 fits)



– Now summarize and extend.

4. HOW WE GET AWAY WITH PQCD

- Specific problems for perturbation theory in QCD

1. Confinement

$$\int e^{-iq \cdot x} \langle 0 | T[f_a(x) \dots] | 0 \rangle$$

has no $q^2 = m^2$ pole for f_a that
transforms nontrivially under color (confinement)

2. The pole at $p^2 = m_\pi^2$

$$\int e^{-iq \cdot x} \langle 0 | T[\pi(x) \dots] | 0 \rangle$$

is not accessible to perturbation theory (χ SB etc., etc.)

- And yet we use infrared safety & asymptotic freedom:

$$\begin{aligned}
 Q^2 \hat{\sigma}_{\text{SD}}(Q^2, \mu^2, \alpha_s(\mu)) &= \sum_n c_n(Q^2/\mu^2) \alpha_s^n(\mu) + \mathcal{O}(1/Q^p) \\
 &= \sum_n c_n(1) \alpha_s^n(Q) + \mathcal{O}(1/Q^p)
 \end{aligned}$$

- What can we really calculate? PT for color singlet operators

– $\int e^{-iq \cdot x} \langle 0 | T[J(x)J(0) \dots] | 0 \rangle$ for color singlet currents

e^+e^- total . . . no QCD in initial state

– Another class of color singlet matrix elements:

$$\lim_{R \rightarrow \infty} R^2 \int dx_0 \int d\hat{n} f(\hat{n}) e^{-iq \cdot y} \langle 0 | J(0) T[\hat{n}_i \Theta_{0i}(x_0, R\hat{n}) J(y)] | 0 \rangle$$

With Θ_{0i} the energy momentum tensor

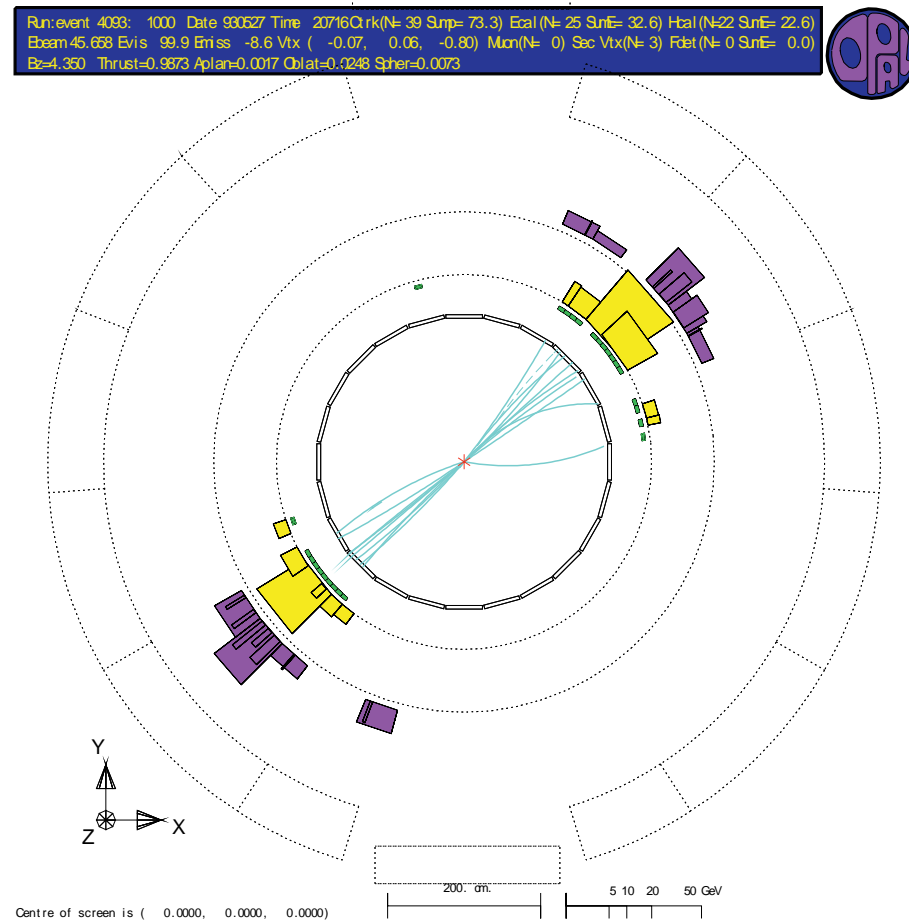
“Weight” $f(\hat{n})$ introduces no new dimensional scale

Short-distance dominated if all $d^k f / d\hat{n}^k$ bounded

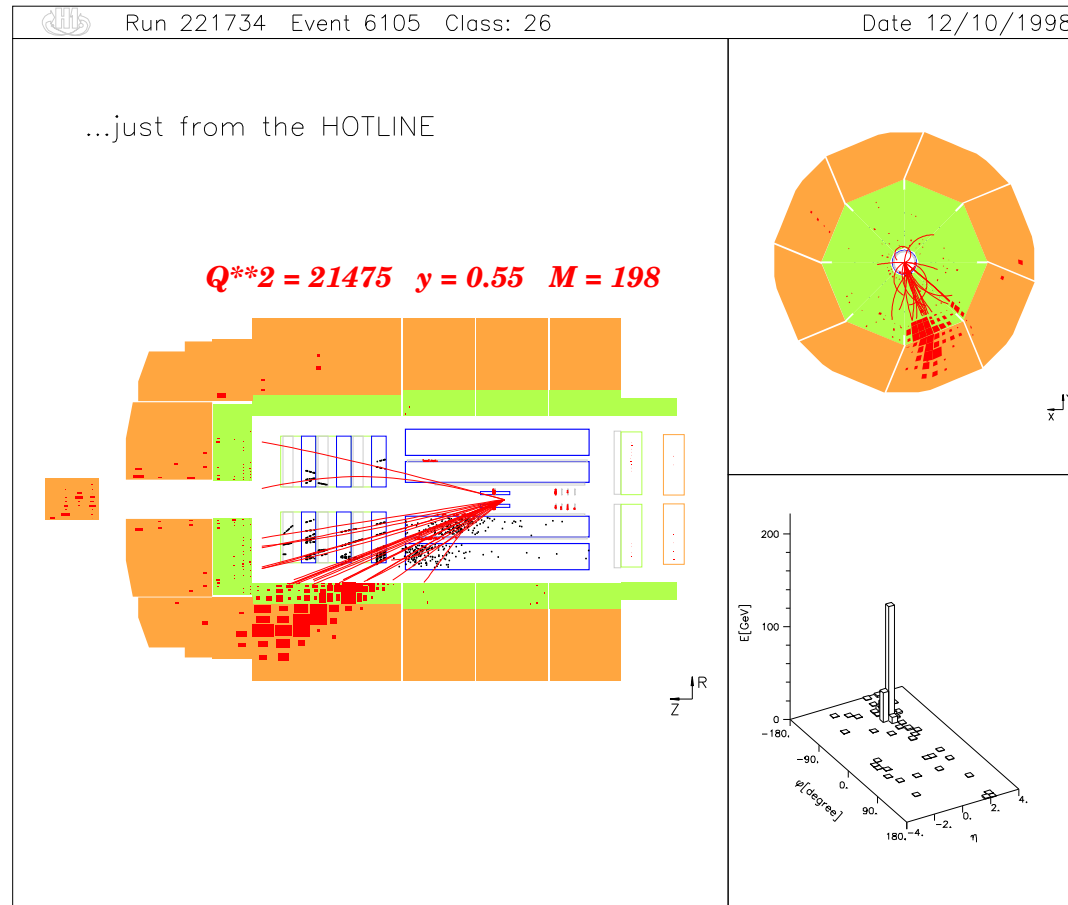
Individual final states have IR divergences, but these cancel in sum over collinear splitting/merging and soft parton emission because they respect energy flow

- The essence of jet computability

- For e^+e^- :

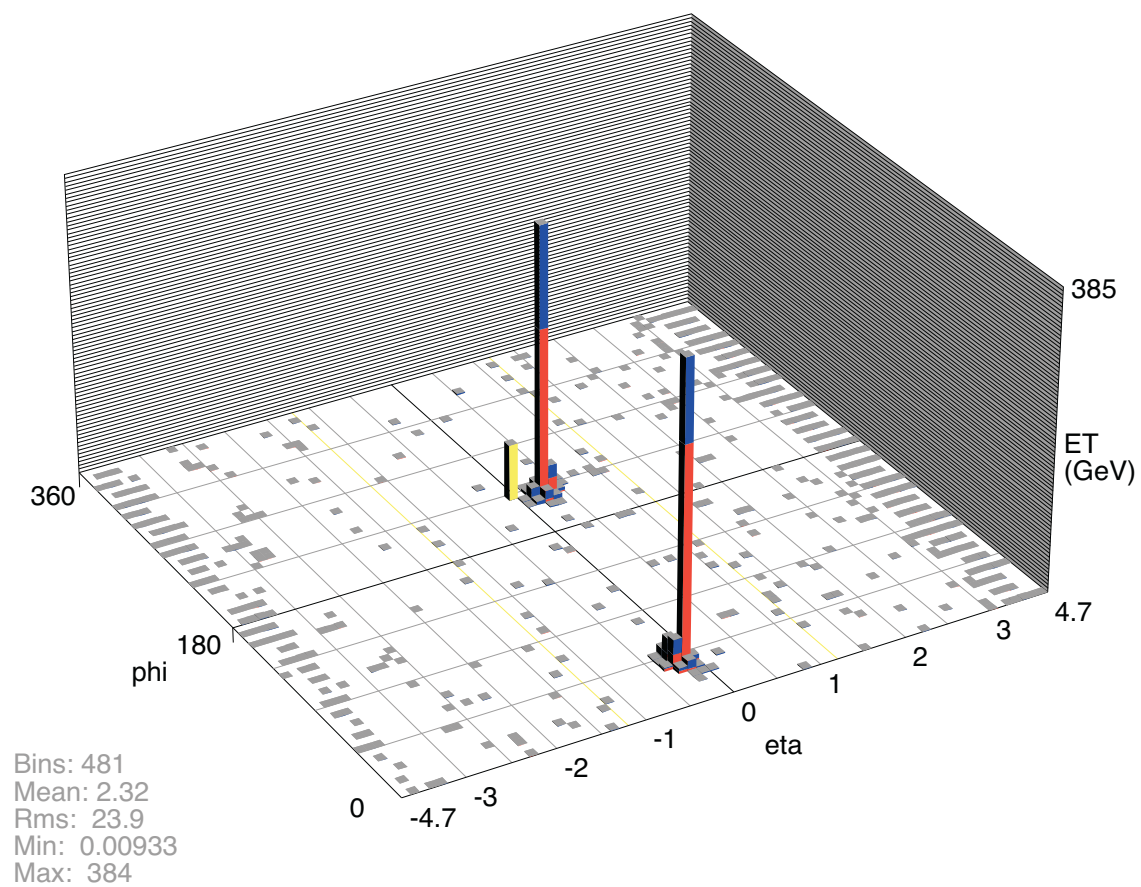


- And for DIS:



- And in nucleon-nucleon collisions

Run 178796 Event 67972991 Fri Feb 27 08:34:03 2004



mE_t : 72.1
 ϕ_t : 223 deg

But what of the initial state? (viz. parton model)

- Factorization

$$Q^2 \sigma_{\text{phys}}(Q, m) = \omega_{\text{SD}}(Q/\mu, \alpha_s(\mu)) \otimes f_{\text{LD}}(\mu, m) + \mathcal{O}(1/Q^p)$$

- μ = factorization scale; m = IR scale (m may be perturbative)
- New physics in ω_{SD} ; $f_{\text{LD}} = f$ and/or D “universal”
- ep DIS inclusive, $pp \rightarrow \text{jets}$, $Q\bar{Q}$, $\pi(p_T)$. . .
- Exclusive limits: $e^+e^- \rightarrow JJ$ as $m_J \rightarrow 0$

- Whenever there is factorization, there is evolution

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\text{phys}}(Q, m)$$

$$\mu \frac{d \ln(f \text{ or } D)}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

PDF f or Fragmentation D

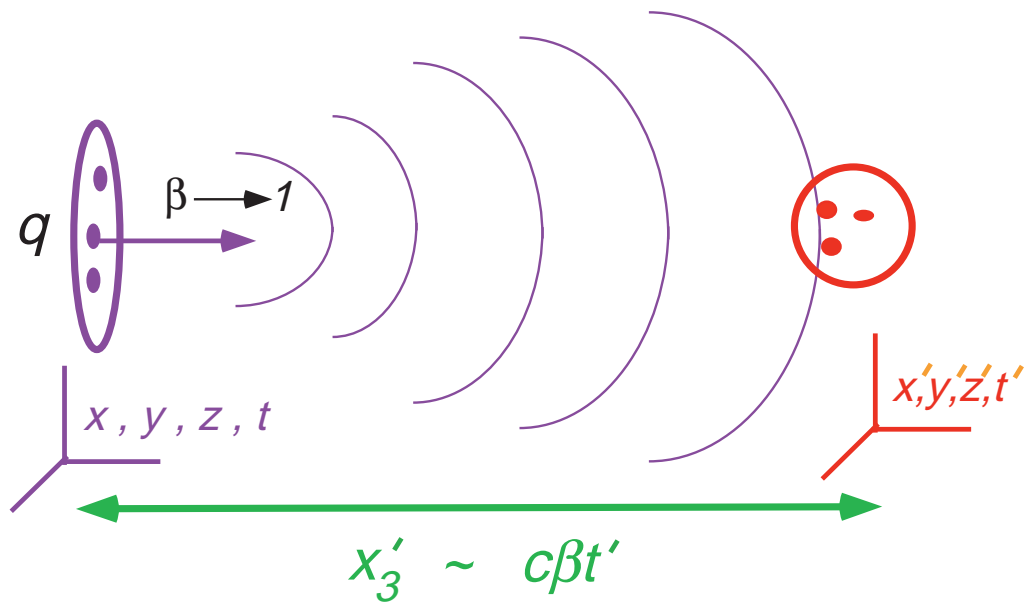
- Wherever there is evolution there is resummation

$$\ln \sigma_{\text{phys}}(Q, m) = \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P(\alpha_s(\mu')) \right\}$$

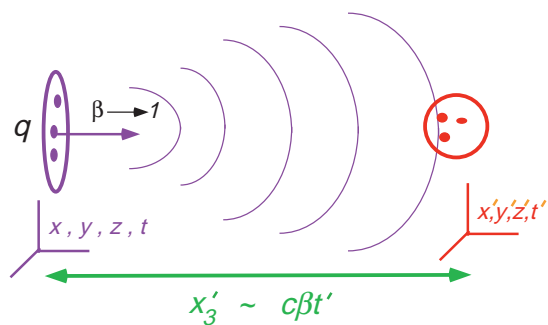
- Factorization proofs:

- (1) ω_{SD} incoherent with long-distance dynamics
- (2) Mutual incoherence when $v_{\text{rel}} = c$:
Jet-jet factorization Ward identities.
- (3) Wide-angle soft radiation sees only total color flow:
jet-soft factorization Ward identities: Wilson lines.
- (4) Dimensionless coupling and renormalizability
 \Leftrightarrow no worse than logarithmic divergence in the IR:
fractional power suppression \Rightarrow finiteness

* **Why Factorization?** Heuristic, classical argument:



$$\Delta \equiv \beta c t' - x'_3$$



<u>field</u>	<u>x frame</u>	<u>x' frame</u>
scalar	$\frac{q}{ \vec{x} }$	$\frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$
gauge	$A_0(x) = \frac{q}{ \vec{x} }$	$A'_0(x') = \frac{q\gamma\Delta}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{-q}{ \vec{x} ^2}$	$E'_3(x') = \frac{-q\gamma\Delta}{(x_T^2 + \gamma^2 \Delta^2)^{3/2}}$

- **Classical: Lorentz contracted fields of incident particles don't overlap until the moment of the scattering, creation of heavy particle, etc.!**
- **Initial-state interactions decouple from the hard process**
- Summarized by multiplicative factors:
parton distributions
- Evolution of partons to jets/hadrons too late to know details of the hard scattering
- Summarized by multiplicative factors:
fragmentation functions
- “Left-over” cross section for hard scattering is IR safe

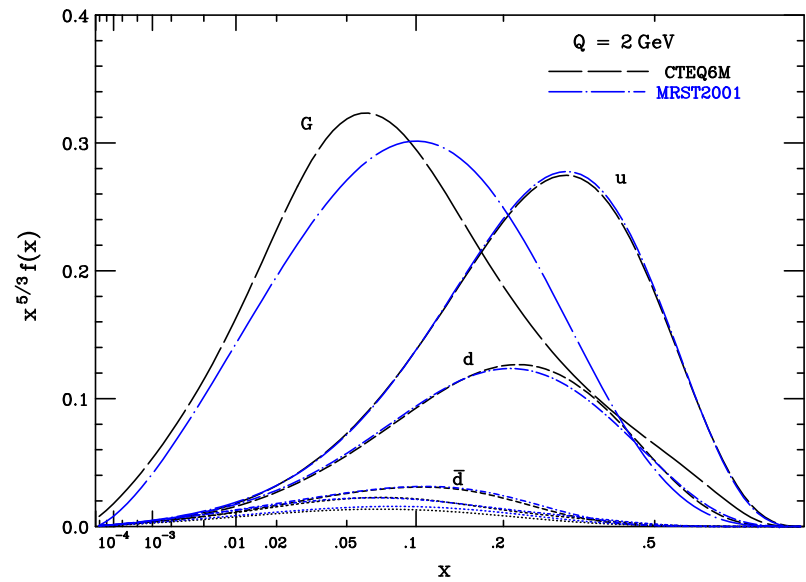
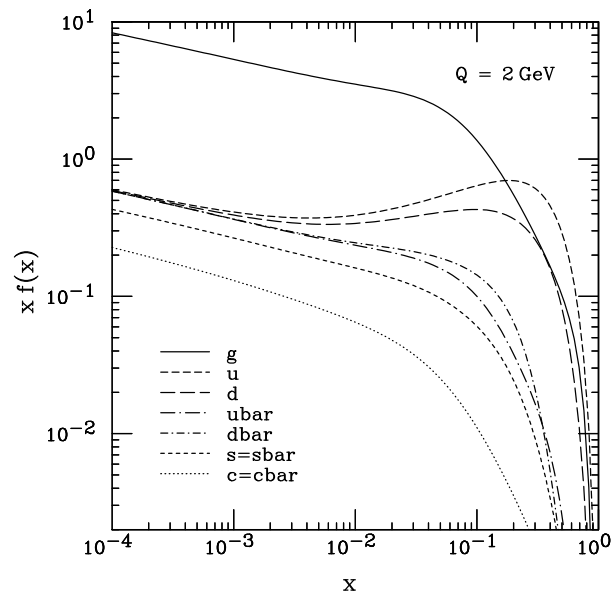
5. INCLUSIVE EW ANNIHILATION IN PQCD

$$\frac{d\sigma_{NN \rightarrow \mu^+ \mu^- + X}(Q, p_1, p_2)}{dQ^2} = \int_{\xi_1, \xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a} \rightarrow \mu^+ \mu^- (Q) + X}(Q, \mu, \xi_1 p_1, \xi_2 p_2)}{dQ^2} \times f_{a/N}(\xi_1, \mu) f_{\bar{a}/N}(\xi_2, \mu)$$

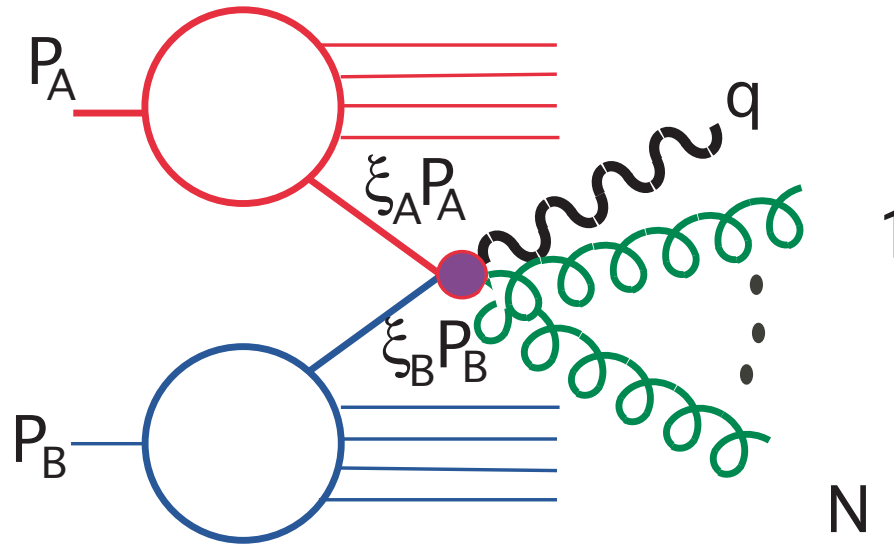
- μ is the factorization scale: separates IR from UV in quantum corrections. μ appears in $\hat{\sigma}$, as $\alpha_s(\mu)$ and as $\ln(\mu/Q)$ so choosing $\mu \sim Q$ can improve perturbative predictions
- Evolution: $\mu df(x, \mu)/d\mu = \int_x^1 P(x/\xi) f(\xi, \mu)$ makes energy extrapolations possible.

Two portraits of modern parton distributions

- * CTEQ6 as seen at moderate momentum transfer:
- * Two modern fits compared (note weighting with x)



- The factorized picture



sum $N = 0$ (PM) to infinity

- High- p_T radiation “has a place to go.”
The rest ($p_T < \mu$) to the PDFs.

6. USING PQCD CORRECTIONS

The transverse momentum distribution at order α_s

Extend factorization to gluon radiation process:

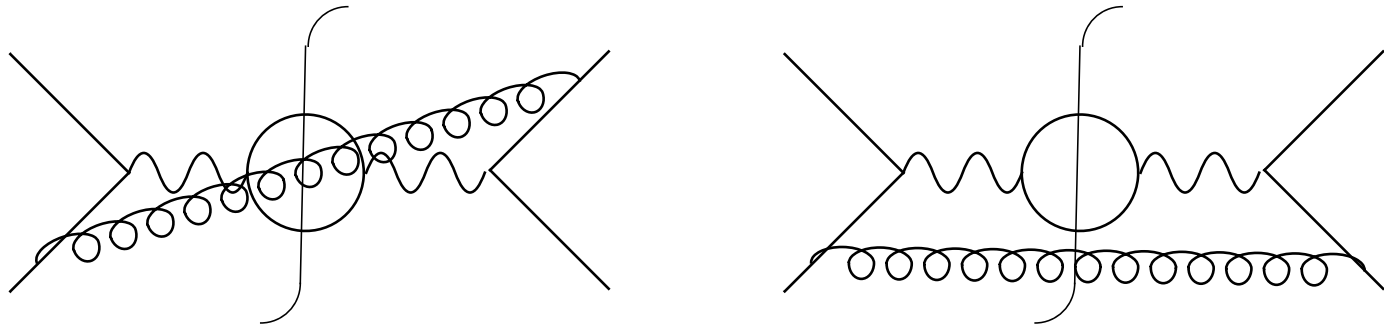
$$q(p_1) + \bar{q}(p_2) \rightarrow \gamma^*(Q) + g(k) ,$$

Treat this $2 \rightarrow 2$ process at lowest order (α_s) “LO”
in factorized cross section, so that $k = -Q$

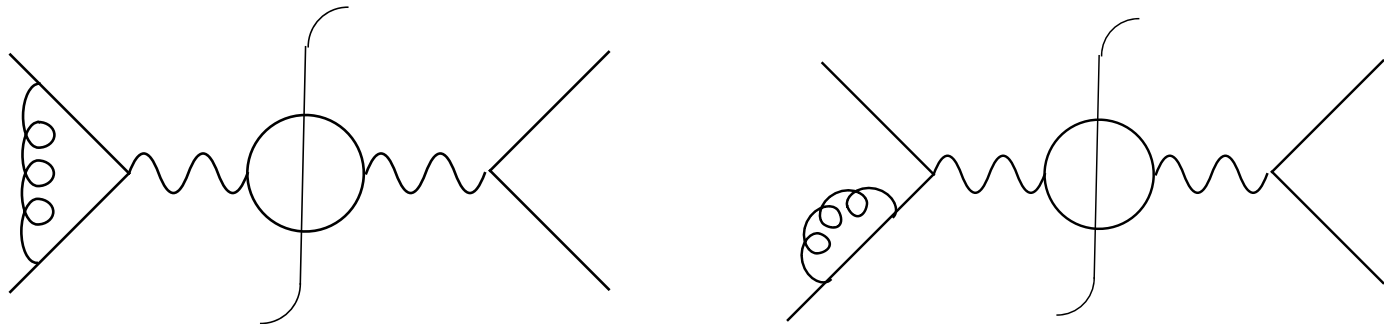
The result is well-defined for $Q_T \neq 0$

- The diagrams at order α_s

Gluon emission contributes at $Q_T \neq 0$



Virtual corrections contribute only at $Q_T = 0$



$$\frac{d^2\sigma_{q\bar{q}\rightarrow\gamma^*g}^{(1)}(z, Q^2, \mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \hat{s}}\right)^{-1/2} \times \left[\frac{1}{\mathbf{Q}_T^2} \frac{1+z^2}{(1-z)} - \frac{2z}{(1-z)Q^2} \right]$$

Fine as long as $\mathbf{Q}_T \neq 0$, $z = Q^2/S \neq 1$.

Q_T integral $\rightarrow \ln(1-z)/(1-z)$, z integral $\rightarrow \ln(Q_T)/Q_T$.

Both off these limits can be dealt with by reorganization, “resummation” of higher order corrections

- **Fundamental application: the total cross section**

Integrate over Q_T at fixed $z = Q^2/S$. $Q_T \rightarrow 0$ is singular

Add diagrams with virtual gluons: *their* Q_T integrals are singular

Remove (factor) low $k_T = -Q_T < \mu$ gluons

The remainder is now finite at fixed Q_T , $z \neq 1$. Combine with LO

But the left-over NLO $\hat{\sigma}$ is not a normal function of z !

**Because $d\sigma/dQ^2$ begins at α_s^0 ,
this is next-to-leading order (NLO) here**

- $\hat{\sigma}_{\bar{q}q}$ for Drell-Yan at NLO

$$\frac{d^2 \hat{\sigma}_{q\bar{q} \rightarrow \gamma^* g}^{(1)}(z, Q^2, \mu^2)}{dQ^2} = \sigma_0(Q^2) \left(\frac{\alpha_s(\mu)}{\pi} \right) \left\{ 2(1+z^2) \left[\frac{\ln(1+z^2)}{1-z} \right]_+ - \frac{[(1+z^2) \ln z]}{(1-z)} + \left(\frac{\pi^2}{3} - 4 \right) \delta(1-z) \right\} + \sigma_0(Q^2) C_F \frac{\alpha_s}{\pi} \left[\frac{1+z^2}{1-z} \right]_+ \ln \left(\frac{Q^2}{\mu^2} \right)$$

- Plus distributions: “generalized functions” (c.f. delta function)
- μ -dependence: evolution for hadron-hadron scattering

- What they are, how they work

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}$$

$$\int_0^1 dx f(x) \left(\frac{\ln(1-x)}{1-x} \right)_+ \equiv \int_0^1 dx (f(x) - f(1)) \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where $f(x)$ will be parton distributions

- $f(x)$ term: real gluon, with momentum fraction $1-x$
- $f(1)$ term: virtual, with elastic kinematics

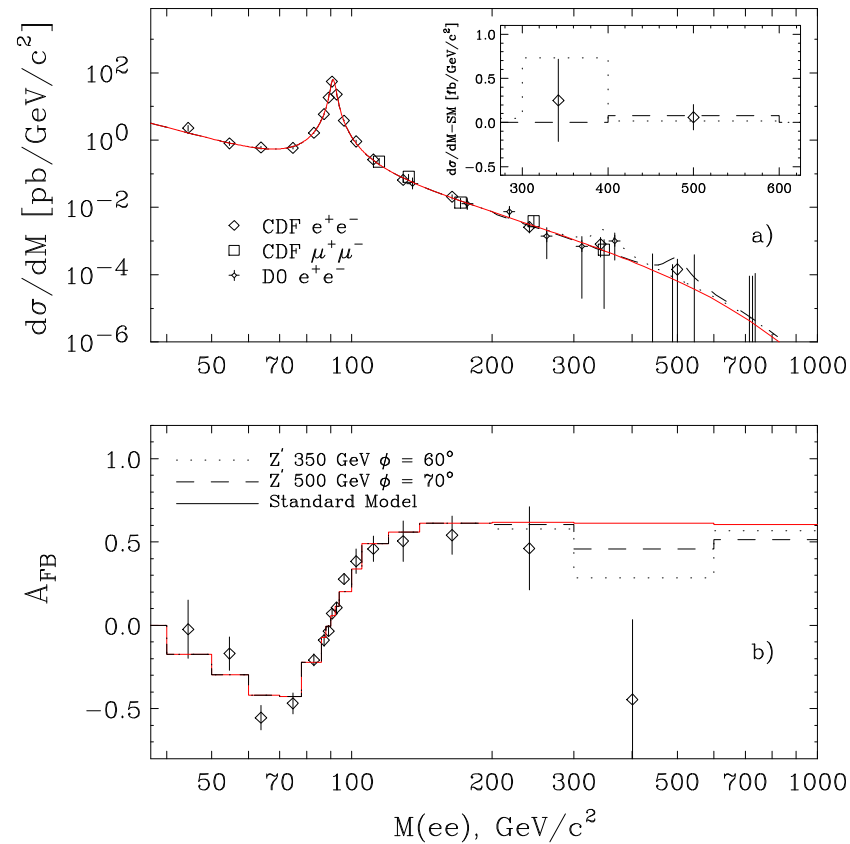
- *A Special Distribution*
- *DGLAP “evolution kernel” = “splitting function”*

$$P_{qq}(x) = C_F \frac{\alpha_s}{\pi} \left[\frac{1+x^2}{1-x} \right]_+$$

- **Nonsinglet, leading order**

Applications

- M-dependence for dileptons at high energy (γ and Z) & forward-backward asymmetry in σ_{Born} compared to NLO
- A test for “new” physics in the hard scattering**



7. GETTING THE PDFs FROM DATA

W asymmetries at the Tevatron: d/u

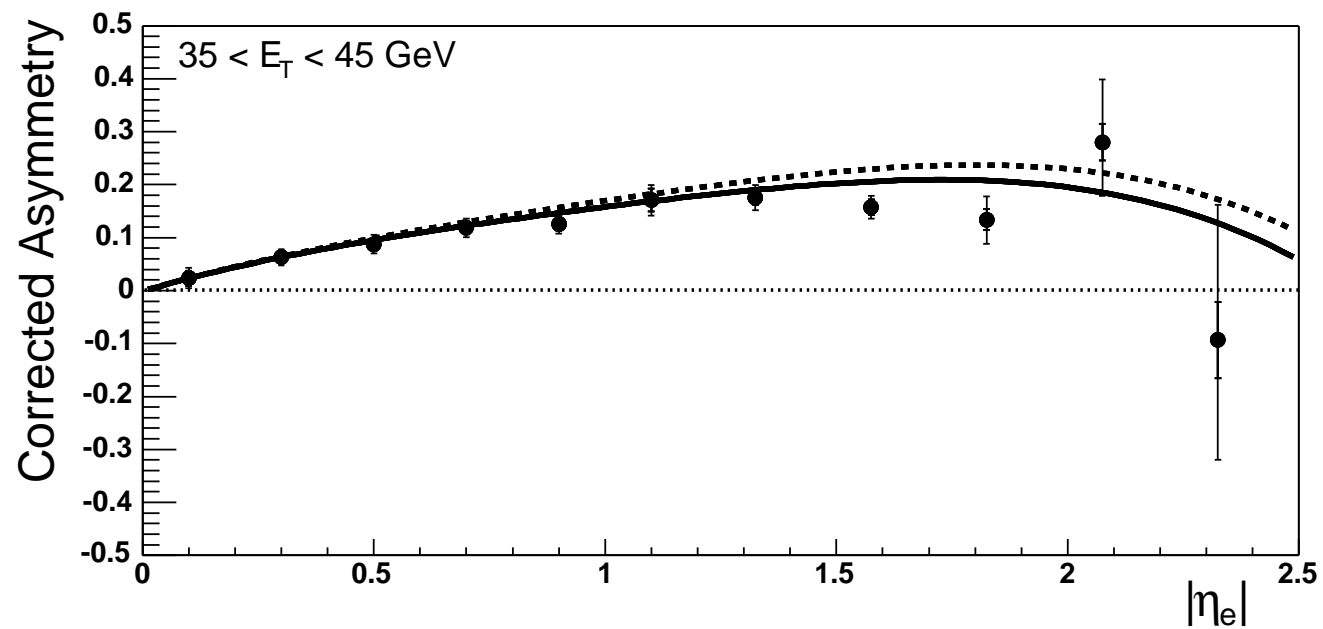
W^+ requires $u\bar{d}$, W^- needs $\bar{u}d$

At LO, since $u_p = \bar{u}_{\bar{p}}$, etc.

$$\frac{d\sigma_{W^+}}{d\eta} = \frac{2\pi G_F}{\sqrt{2}} u_p(x_a = \sqrt{\tau}e^\eta) d_p(x_b = \sqrt{\tau}e^{-\eta})$$

Asymmetry tests d/u as a function of

$$A(y) \equiv \frac{\sigma_{W^+}(\eta) - \sigma_{W^-}(\eta)}{\sigma_{W^+}(\eta) + \sigma_{W^-}(\eta)} = \frac{u_p(x_a) d_p(x_b) - d_p(x_a) u_p(x_b)}{u_p(x_a) d_p(x_b) + d_p(x_a) u_p(x_b)}$$



(CDF Collaboration, Phys. Rev. D71, 051104 (2005) hep-ex/0501023)

- **Foward fixed target DY ($\tau = M^2/S$) and \bar{d}/\bar{u}**

At LO,

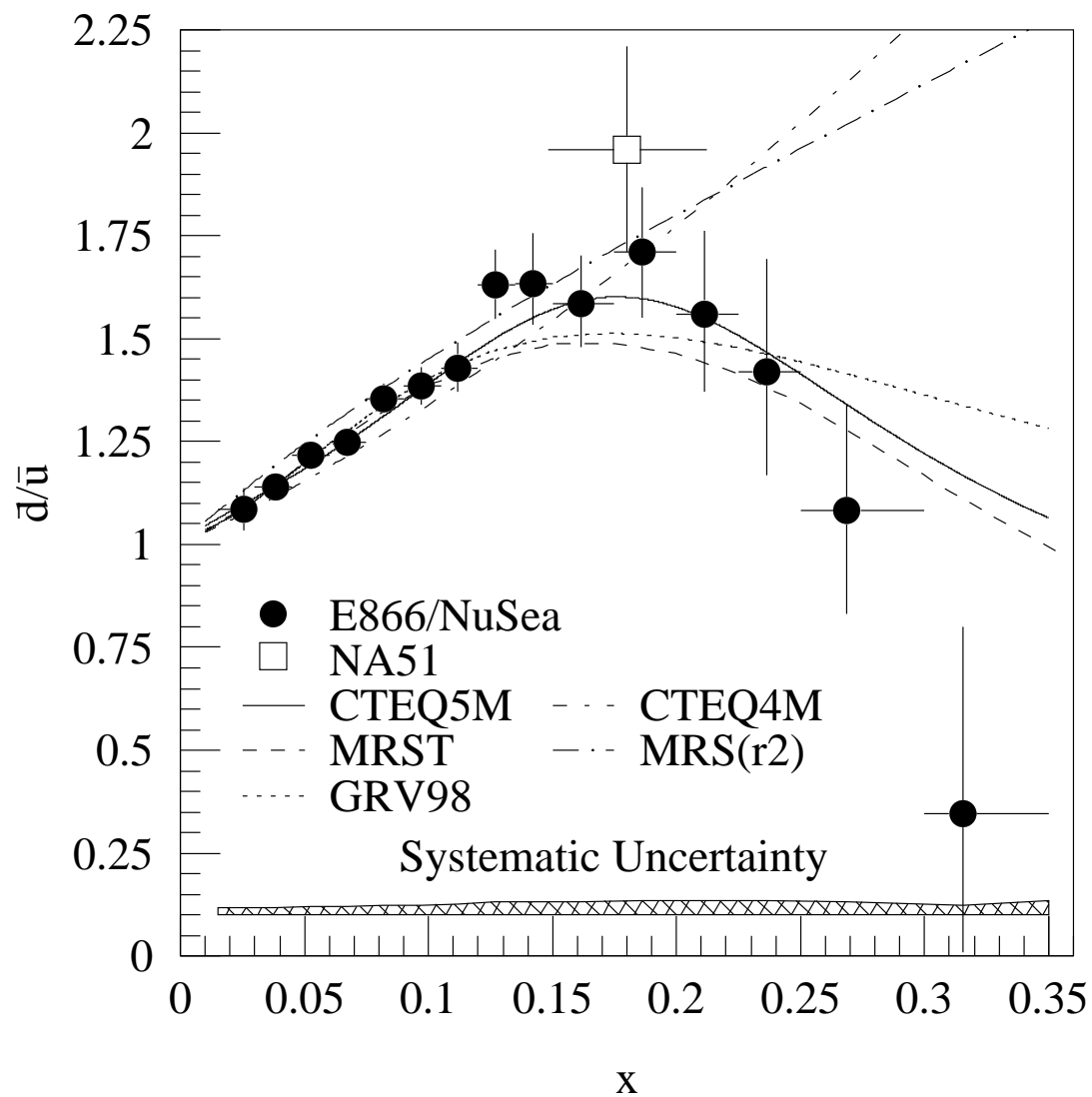
$$\frac{d\sigma_{pN}}{dM^2 d\eta} = \left(\frac{4\pi\alpha_{\text{EM}}^2}{9M^4} \right) \sum_a \lambda_a^2 f_{a/p}(\sqrt{\tau}e^\eta, M) f_{\bar{a}/N}(\sqrt{\tau}e^{-\eta}, M)$$

Large η ; a valence, \bar{a} sea: sensitivity to sea distribution

E866: compare pp and pd

$$\frac{\sigma_{pD}}{2\sigma_{pp}} \sim \frac{1}{2} \left(1 + \frac{\bar{d}_p(\sqrt{\tau}e^{-\eta})}{\bar{u}_p(\sqrt{\tau}e^{-\eta})} \right)$$

Previously unavailable information on the sea ratio



E866/NuSea Collaboration, Phys. Rev. D64, 052002 (2001) hep-ex/0103030

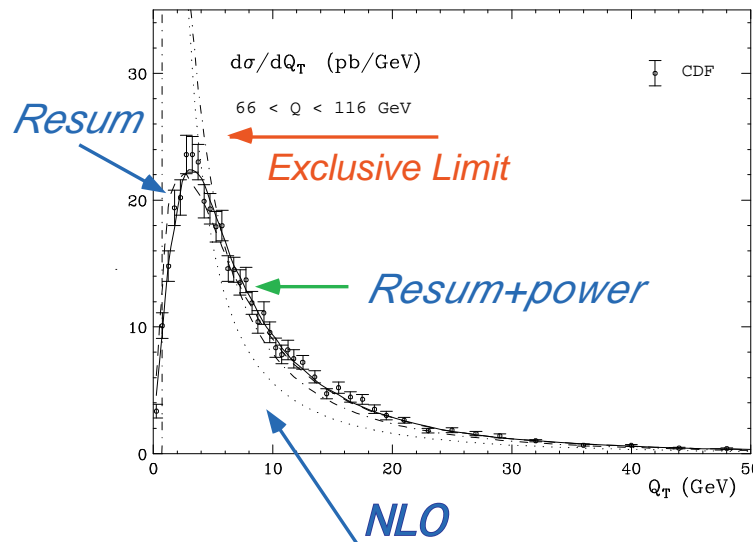
8. USING RESUMMATION: THE Q_T DISTRIBUTION

- **Low Q_T Drell-Yan & Higgs at leading log (LL)** ($\alpha_s^n \ln^{2n-1} Q_T$)

$$\frac{d\sigma(Q)}{dQ_T} \sim \frac{d}{dQ_T} \exp \left[-\frac{\alpha_s}{\pi} C_F \ln^2 \left(\frac{Q}{Q_T} \right) \right]$$

($C_F = 4/3$)

- **Double jet-soft factorization \rightarrow double logs** (from A. Kulesza, G.S., W. Vogelsaar)



- **General features:**

**Maximum then decrease near “exclusive” limit
(parton model kinematics) replaces divergence at $Q_T = 0$**

Soft but perturbative radiation broadens distribution

**Typically nonperturbative correction necessary for
full quantitative description**

Recover fixed order predictions $\sigma^{(1)}$ away from exclusive limit

**Generally requires (Fourier) transform (impact parameter)
to go beyond leading log**

Getting to $Q_T \ll Q$: Transverse momentum resummation

- (Logs of Q_T)/ Q_T to all orders

How? Variant factorization and separation of variables

q and \bar{q} “arrive” at point of annihilation with transverse momentum of radiated gluons in initial state.

q and \bar{q} radiate independently (fields don’t overlap!).

Final-state QCD radiation too late to affect cross section

Summarized by: Q_T -factorization:

$$\begin{aligned} \frac{d\sigma_{NN \rightarrow QX}}{dQ d^2Q_T} &= \int d\xi_1 d\xi_2 d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}_{sT} \delta(Q_T - k_{1T} - k_{2T} - k_{sT}) \\ &\times H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a} \rightarrow Q+X} \\ &\times \mathcal{P}_{a/N}(\xi_1, p_1 \cdot n, k_{1T}) \mathcal{P}_{\bar{a}/N}(\xi_2, p_2 \cdot n, k_{2T}) U_{a\bar{a}}(k_{sT}, n) \end{aligned}$$

The \mathcal{P}'_s : new Transverse momentum-dependent PDFs

Also need U : “soft function” for wide-angle radiation

- Symbolically:

$$\frac{d\sigma_{NN \rightarrow QX}}{dQ d^2Q_T} \quad H \times \mathcal{P}_{a/N}(\xi_1, p_1 \cdot n, k_{1T}) \mathcal{P}_{\bar{a}/N}(\xi_2, p_2 \cdot n, k_{2T}) \\ \otimes_{\xi_i, k_{iT}} U_{a\bar{a}}(k_{sT}, n)$$

We will solve for the k_T dependence of the \mathcal{P} 's.

New factorization variables: n^μ apporitions gluons k :

$$p_i \cdot k < n \cdot k \Rightarrow k \in \mathcal{P}_i$$

$$p_a \cdot k, p_{\bar{a}} \cdot k > n \cdot k \Rightarrow k \in U$$

Convolution in $k_{i,T}$ s \Rightarrow Fourier $e^{i\vec{Q}_T \cdot \vec{b}}$

- The factorized cross section in “impact parameter space”:

$$\begin{aligned} \frac{d\sigma_{NN \rightarrow QX}(Q, b)}{dQ} &= \int d\xi_1 d\xi_2 \\ &\times H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a} \rightarrow Q+X} \\ &\times \mathcal{P}_{a/N}(\xi_1, p_1 \cdot n, b) \mathcal{P}_{\bar{a}/N}(\xi_2, p_2 \cdot n, b) U_{a\bar{a}}(b, n) \end{aligned}$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\text{ren}}, n \Rightarrow$ two equations

$$\mu_{\text{ren}} \frac{d\sigma}{d\mu_{\text{ren}}} = 0 \quad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

- **Solve and transform back to Q_T : all the (Logs of Q_T)/ Q_T :**

$$\frac{d\sigma_{NN\text{res}}}{dQ^2 d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2} e^{i\vec{Q}_T \cdot \vec{b}} \exp [E_{a\bar{a}}^{\text{PT}}(b, Q, \mu)]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{d\hat{\sigma}_{a\bar{a} \rightarrow \mu^+ \mu^- (Q) + X}(Q, \mu)}{dQ^2} f_{a/N}(\xi_1, 1/b) f_{\bar{a}/N}(\xi_2, 1/b)$$

“Sudakov” exponent suppresses large $b \leftrightarrow$ small Q_T :

$$E_{a\bar{a}}^{\text{PT}} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) + 2B_q(\alpha_s(k_T)) \right]$$

With $B = 2(K + G)_{\mu=p \cdot n}$, and lower limit: $1/b$ (NLL)

- **Leading log: fixed $\alpha_s(Q)$, $A^{(1)}(\alpha_s/\pi)$ only**

$$\frac{d\sigma_{NN\text{res}}}{dQ^2 d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2} e^{i\vec{Q}_T \cdot \vec{b}} \exp \left[- A^{(1)}(\alpha_s(Q)/\pi) \ln^2(bQ) \right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{d\hat{\sigma}_{a\bar{a} \rightarrow \mu^+ \mu^- (Q) + X}(Q, \mu)}{dQ^2} f_{a/N}(\xi_1, 1/b) f_{\bar{a}/N}(\xi_2, 1/b)$$

- **If ignore evolution of the f 's, get an overall factor**

$$\frac{d\sigma_{NN \rightarrow \mu^+ \mu^- + X}(Q, \mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\partial}{\partial Q_T^2} e^{-\left[A^{(1)}(\alpha_s(Q)/\pi) \ln^2(Q^2/Q_T^2) \right]}$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{\hat{\sigma}_{a\bar{a} \rightarrow \mu^+ \mu^- (Q) + X}(Q, \mu)}{dQ^2} f_{a/N}(\xi_1, \mu) f_{\bar{a}/N}(\xi_2, \mu)$$

– **Comments:**

The functions $A_i(\alpha_s)$ and $B_i(\alpha_s)$ are anomalous dimensions.

And can be calculated by comparison to low orders.

In particular, $A_i(\alpha_s)$ is the numerator of the $1/(1-x)$ term in splitting function $P_{ii}(x)$

because it's the **infrared divergent** ($x \rightarrow 1$) **coefficient** of **the collinear** $b \rightarrow \infty$ singularity.

$$- A_q(\alpha_s) = \frac{\alpha_s}{\pi} C_q \left(1 + \frac{\alpha_s}{\pi} K + \dots \right), \quad K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5n_F}{9}$$

– **Logs from LO, NLO in** $A_q = A_q^{(1)}(\alpha_s/\pi) + \dots$, **LO in** B_q

$$E_{q\bar{q}} = -2 \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) + B(\alpha_s(k_T)) \right]$$

$$\sim 2C_i \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[\left\{ \frac{\alpha_s(k_T)}{\pi} + K \frac{\alpha_s(k_T)}{\pi} \right\} \ln \left(\frac{Q^2}{k_T^2} \right) + 2 \frac{\alpha_s(k_T)}{\pi} \right]$$

$$\sim 2C_i \frac{\alpha_s(Q)}{\pi} \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[\left\{ 1 + \left(\frac{\alpha_s(Q)}{\pi} \right) (K - \beta_0) \right\} \ln \left(\frac{Q^2}{k_T^2} \right) + 2 \frac{\alpha_s(Q)}{\pi} \right]$$

– **The pattern:**

$$\begin{aligned}
& 2C_i \frac{\alpha_s(Q)}{\pi} \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[\left\{ 1 + \left(\frac{\alpha_s(Q)}{\pi} \right) \left(K - \frac{\beta_0}{4\pi} \right) \right\} \ln \left(\frac{Q^2}{k_T^2} \right) \right. \\
& \quad \left. + 2 \frac{\alpha_s(Q)}{\pi} \right] \\
& \sim \alpha_s \ln^2(bQ) (1 + \alpha_s \ln(bQ) + \dots) \\
& \quad + \alpha_s \ln(bQ) (1 + \alpha_s \ln(bQ) + \dots) \\
& \quad + \dots
\end{aligned}$$

– **These are LL($A^{(1)}$), NLL ($B^{(1)}$, $A^{(2)}$), and so on**

– **NLL is good so long as $\alpha_s(Q) \ln bQ \leq 1$.**

- Evaluating a resummed cross sections: re-enter NPQCD.

We start with:

$$E^{\text{PT}} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) + B_q(\alpha_s(k_T)) \right]$$

With running coupling:

$$\alpha_s(k_T) = \frac{\alpha_s(Q)}{1 + \frac{\alpha_s(Q)}{4\pi} \beta_0 \ln \left(\frac{k_T^2}{Q^2} \right)} = \frac{4\pi}{\beta_0 \ln \left(\frac{k_T^2}{\Lambda_{\text{QCD}}^2} \right)}$$

Singularity in integral at $b^2 = Q^2 \exp[-4\pi/\beta_0\alpha_s(Q)] \sim \frac{1}{\Lambda^2}$.

– Problem: how to do the inverse transform with the running coupling when $k_T^{\min} \sim 1/b$ gets small?

– At least four approaches:

1) Work in Q_T -space directly to some approximation

The originals: Dokshitzer, Diakanov & Troyan

Revived by Ellis & Veseli Kulesza & Stirling

who re-derived it from b -space.

2) Insert a “soft landing” on the k_T integral by replacing

$$1/b \rightarrow \sqrt{1/b^2 + 1/b_*^2}$$

for some fixed b_* . (CS, CSS “ b_* ” prescription, ResBos)

3) Extrapolation of E^{PT} into NP region (Qiu, Zhang).

4) Minimal: avoid the singularity at $1/b = \Lambda_{\text{QCD}}$ by monkeying with the b -space contour integral.
(This technique introduced in threshold resummation; then adapted by Laenen, GS and Vogelsang, and Bozzi, Catani, de Florian and Grazzini.)

Any of these “define” PT. All will fit the data qualitatively, and with a little work quantitatively.

But all require new parameters for quantitative fit.
This is not all bad . . . let's see why.

$$\begin{aligned}
E^{\text{soft}} &= \frac{1}{2\pi} \int_0^{\mu_I^2} \frac{d^2 k_T}{k_T^2} A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) (e^{i\mathbf{b} \cdot \mathbf{k}_T} - 1) \\
&\sim - \int_0^{\mu_I^2} \frac{dk_T^2}{k_T^2} (\mathbf{b} \cdot \mathbf{k}_T)^2 A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) + \dots \\
&\sim - b^2 \int dk_T^2 A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right)
\end{aligned}$$

$\theta(k_T - 1/b) \Rightarrow (e^{i\mathbf{b} \cdot \mathbf{k}_T} - 1)$; **in fact, correct to all orders,**

Note the expansion is for b “small enough” only.

What is $-b^2 \int dk_T^2 A_q(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right)$?

**Don't really know, but it suggests
a nonperturbative correction of the form
(exhibiting the μ_I is unconventional)**

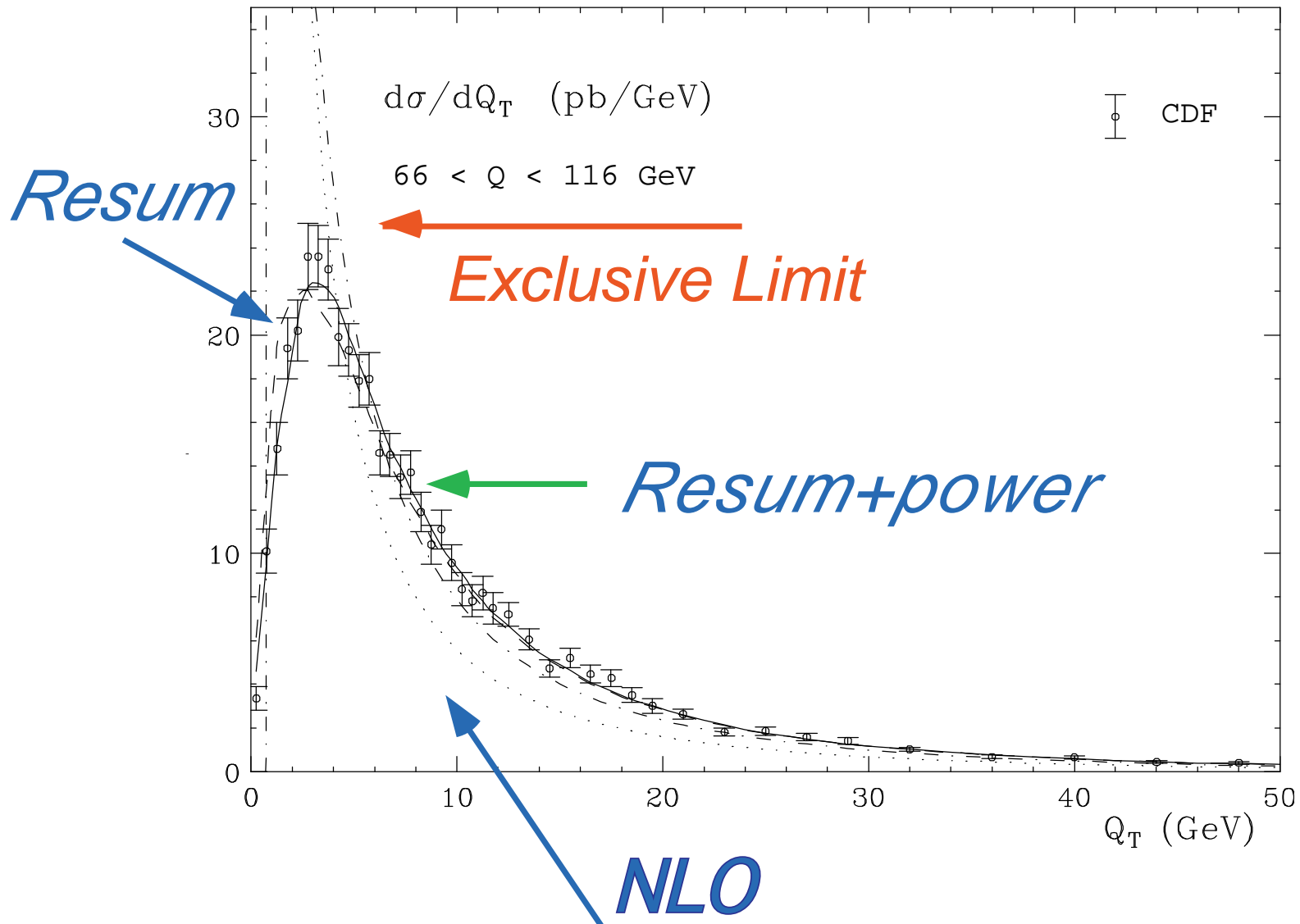
$$E^{\text{NP}} = -b^2 \mu_I^2 \left(g_1 \ln \left(\frac{Q}{\mu_I} \right) + g_2 \right)$$

**Since this is an exponent, whatever the definition
of the perturbative resummed cross section, it is
smeared with a Gaussian whose width in b (k_T) space
decreases (increases) with $\ln Q$.**

In summary

$$\begin{aligned}
 \frac{d\sigma(Q_T)}{dQ^2 d^2\vec{Q}_T} &= \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2} e^{i\vec{Q}_T \cdot \vec{b}} e^{E_{a\bar{a}}^{\text{PT}}(b, Q, \mu)} e^{-\mu_I^2 b^2 (g_1 \ln\left(\frac{Q}{\mu_I}\right) + g_2)} \\
 &\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{d\hat{\sigma}_{a\bar{a} \rightarrow \mu^+ \mu^- (Q) + X}(Q, \mu)}{dQ^2} f_{a/N}(\xi_1, 1/b) f_{\bar{a}/N}(\xi_2, 1/b) \\
 &= \pi \int d^2\mathbf{k}_T \frac{e^{-k_T^2/4[\mu_I^2(g_2 \ln(Q/k_T) + g_2)]}}{\mu_I^2(g_2 \ln(Q/k_T) + g_2)} \frac{d\sigma_{NN}(\mathbf{Q}_T - \mathbf{k}_T)}{dQ^2 d^2\vec{Q}_T}
 \end{aligned}$$

Which gives curves like the one we saw before.



Successful phenomenology for W and Z.

In principle, can also fit to fixed-target Drell-Yan with the same set of NP parameters.

Qiu and Zhang show that NP corrections are dominant for fixed-target Q^2 .

Next – what about those $1/(1 - z)$ (soft gluon energy) singularities?

– Continue with threshold resummation . . .

9. PUTTING IT ALL TOGETHER: OBSERVED HADRONS

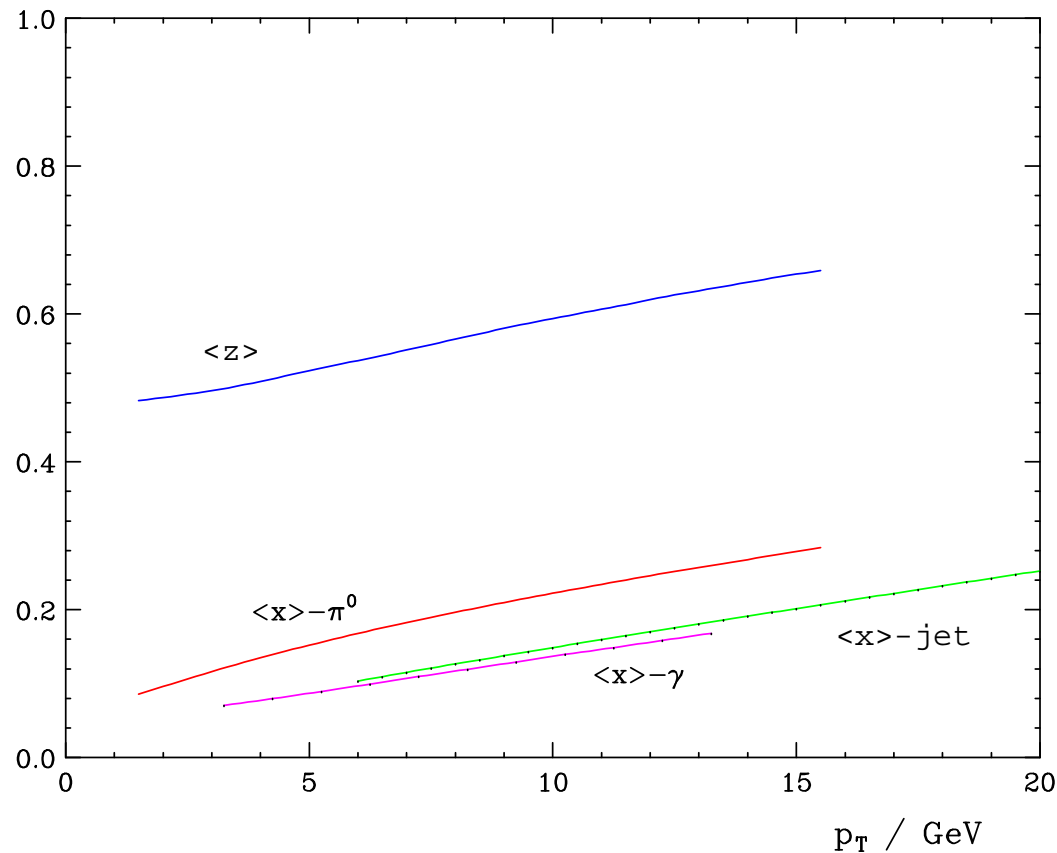
- **Pions at fixed target and RHIC** (Vogelsang and de Florian, 2004)

$$\begin{aligned} \frac{p_T^3 d\sigma(x_T)}{dp_T} &= \sum_{a,b,c} \int_0^1 dx_1 f_{a/H_1}(x_1, \mu_F^2) \int_0^1 dx_2 f_{b/H_2}(x_2, \mu_F^2) \\ &\quad \times \int_0^1 dz z^2 D_{h/c}(z, \mu_F^2) \\ &\quad \times \int_0^1 d\hat{x}_T \delta\left(\hat{x}_T - \frac{x_T}{z\sqrt{x_1 x_2}}\right) \int_{\hat{\eta}_-}^{\hat{\eta}_+} d\hat{\eta} \frac{\hat{x}_T^4 \hat{s}}{2} \frac{d\hat{\sigma}_{ab \rightarrow cX}(\hat{x}_T^2, \hat{\eta})}{d\hat{x}_T^2 d\hat{\eta}} \end{aligned}$$

$\hat{\eta}$: pseudorapidity at parton level

$$\hat{\eta}_+ = -\hat{\eta}_- = \ln \left[(1 + \sqrt{1 - \hat{x}_T^2}) / \hat{x}_T \right]$$

Averages for distribution x and fragmentation z 's



RHIC 200 GeV midrapidity average z for pions, and average x for pions, photons (NLO) Thanks to Werner Vogelsang

– As for the DY Q_T distribution: collinear $f + D + \text{soft} \Rightarrow$ double logs

$$\frac{\hat{s} d\hat{\sigma}_{ab \rightarrow cX}^{(1)}(v, w)}{dv dw} \approx \frac{\hat{s} d\hat{\hat{\sigma}}_{ab \rightarrow cd}^{(0)}(v)}{dv} \left[A' \delta(1-w) + B' \left(\frac{\ln(1-w)}{1-w} \right)_+ + C' \left(\frac{1}{1-w} \right)_+ \right]$$

- 1) For resummation, take x_T^{2N} moments:

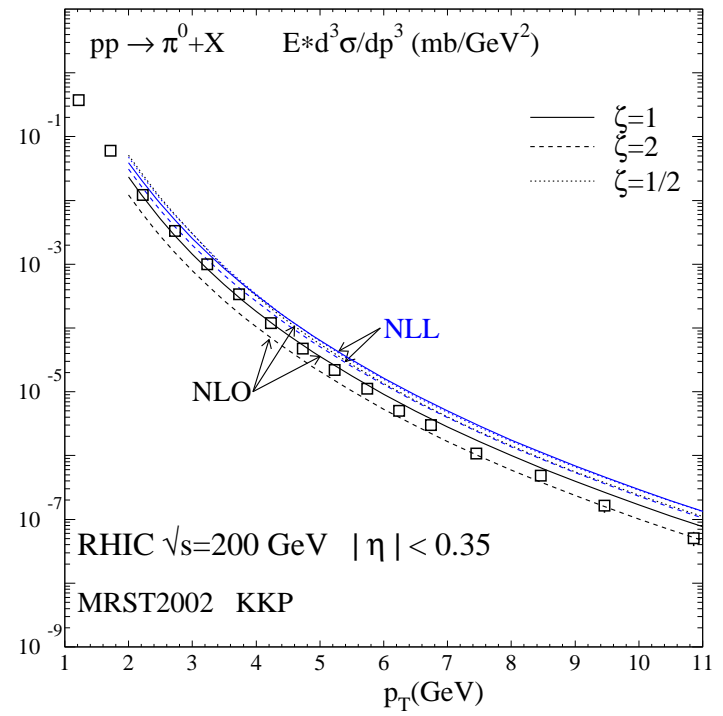
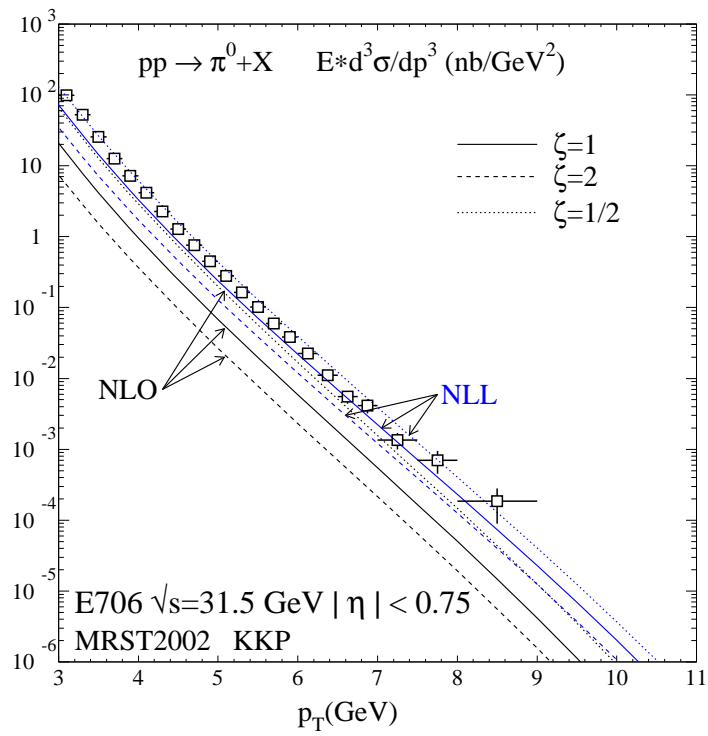
$$\hat{\sigma}_{ab \rightarrow cd}^{(\text{res})}(N) = C_{ab \rightarrow cd} \Delta_N^a \Delta_N^b \Delta_N^c J_N^d \left[\sum_I G_{ab \rightarrow cd}^I \Delta_{I N}^{(\text{int})ab \rightarrow cd} \right] \hat{\sigma}_{ab \rightarrow cd}^{(\text{Born})}(N)$$

- 2) A typical resummed factor

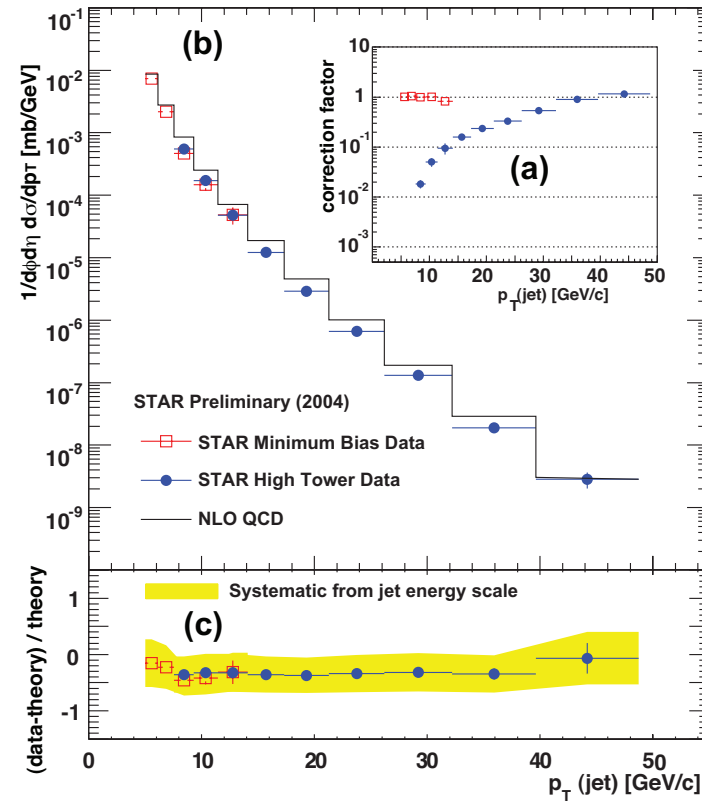
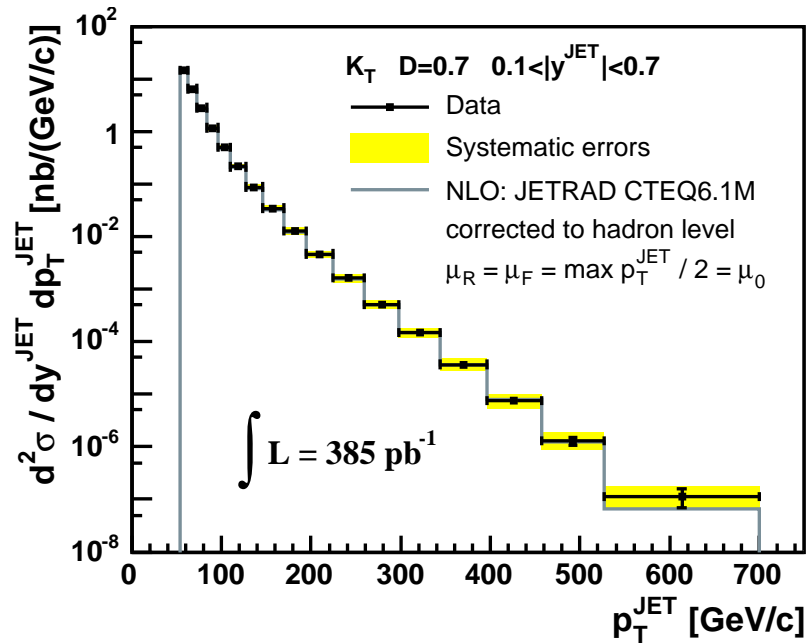
$$\Delta_N^a = \exp \left[\int_0^1 \frac{z^{N-1}-1}{1-z} \int_{\mu_{FI}^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A_a(\alpha_s(q^2)) \right]$$

$$A = C_F(\alpha_s/\pi) + \dots$$

- Invert the moments: resolve a long-standing fixed-target/collider contrast!



- And jets at the Tevatron, and now the RHIC



- Nicely settled down.

Conclusions as Prologue

- pQCD formalism works well in pp collisions
(not least from RHIC data/theory interplay)
- pQCD revolves around factorization and energy flow
- Multiple interactions induce corrections to factorized cross section
typically $(\text{number of partons}) \times (\text{soft scale/hard scale})^2$
- Induced radiation redistributes energy flow
- Centrality in AA collisions a control parameter
for these corrections

- **Heavy quark/quarkonia production of special sensitivity**
- **Nuclear collisions shed light on pQCD and vice-versa**